Lattices

Slides follow Davey and Priestley: *Introduction to Lattices and Order*

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Partial Orders

Let P be a set. A binary relation $\leq$ on P is a partial order iff it is:

1. reflexive: $(\forall x \in P)\, x \leq x$
2. transitive: $(\forall x, y, z \in P)\, x \leq y \wedge y \leq z \implies x \leq z$
3. antisymmetric: $(\forall x, y \in P)\, x \leq y \wedge y \leq x \implies x = y$

An element $\bot$ with $\bot \leq x$ for all $x \in P$ is called bottom element. It is unique. Analogously, $\top$ is called top element, if $\top \geq x$ for all $x \in P$.

Duality

Let P an ordered set. The dual P^D of P is obtained by defining $x \leq y$ in P^D whenever $y \leq x$ in P.

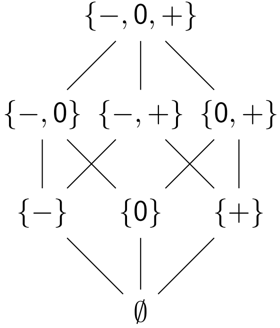
For every statement Φ about P there is a dual statement Φ^D about P^D. It is obtained from P by exchanging $\leq$ by $\geq$.

If Φ is true for all ordered sets, Φ^D is also true for all ordered sets.

Hasse Diagrams

A partial order $(P, \leq)$ is typically visualized by a Hasse diagram:

- Elements of P are points in the plane
- If $x \leq z$, then z is drawn above x.
- If $x \leq z$, and there is no y with $x \leq y \leq z$, then x and z are connected by a line



The Hasse diagram of the dual of P is obtained by “turning” the one of P by 180°

Upper and Lower Bounds

Let $(P, \leq)$ be a partial ordered set and let $S \subseteq P$. An element $x \in P$ is a lower bound of S, if $x \leq s$ for all $s \in S$. Let

$$S^\ell = \{x \in P \mid (\forall s \in S)\, x \leq s\}$$

be the set of all lower bounds of the set S. Dually:

$$S^u = \{x \in P \mid (\forall s \in S)\, x \geq s\}$$

Note: $\emptyset^u = \emptyset^\ell = P$.

If S^ℓ has a greatest element, this element is called the greatest lower bound and is written $\inf S$. (Dually for least upper bound and $\sup S$.) The greatest lower bound only exists, iff there is a $x \in P$ such that

$$(\forall y \in P)\, (((\forall s \in S)\, s \geq y) \iff x \geq y)$$

Complete Partial Orders

A non-empty subset $S \subseteq P$ is directed if for every $x, y \in S$ there is $z \in S$ such that $z \in \{x, y\}^u$.

P is a complete partial order (CPO) if every directed set M has a least upper bound.

We use the notation $\bigsqcup M$ to indicate the least upper bound of a directed set.

Lattices

The order-theoretic definition

Let P be an ordered set.

- If $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for every pair $x, y \in P$ then P is called a lattice.
- If for every $S \subseteq P$, $\sup S$ and $\inf S$ exist, then P is called a complete lattice.

The Connecting Lemma

Let L be a lattice and let $a, b \in L$. The following statements are equivalent:

1. $a \leq b$
2. $\inf\{a, b\} = a$
3. $\sup\{a, b\} = b$

Lattices

The algebraic definition

We now view L as an algebraic structure $(L; \vee, \wedge)$ with two binary operators

$$x \vee y := \sup\{x, y\} \qquad x \wedge y := \inf\{x, y\}$$

Theorem: $\vee$ and $\wedge$ satisfy for all $a, b, c \in L$:

$$\begin{array}{lll}
(L1) & (a \vee b) \vee c = a \vee (b \vee c) & \text{associativity} \\
(L1)^D & (a \wedge b) \wedge c = a \wedge (b \wedge c) & \\
(L2) & a \vee b = b \vee a & \text{commutativity} \\
(L2)^D & a \wedge b = b \wedge a & \\
(L3) & a \vee a = a & \text{idempotency} \\
(L3)^D & a \wedge a = a & \\
(L4) & a \vee (a \wedge b) = a & \text{absorption} \\
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\end{array}$$

Lattices

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(L3) & a \vee a = a & \text{idempotency} \\
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(L4) & a \vee (a \wedge b) = a & \text{absorption} \\
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\end{array}$$

Proof: (L2) is immediate because $\sup\{x, y\} = \sup\{y, x\}$. (L3), (L4) follow from the connection lemma. (L1) exercise. The dual laws come by duality.

Lattices

From the algebraic to the order-theoretic definition

Let $(L; \vee, \wedge)$ be a set with two operators satisfying $(L1)$–$(L4)$ and $(L1)^D$–$(L4)^D$

Theorem:

1. Define $a \leq b$ on L if $a \vee b = b$. Then, $\leq$ is a partial oder
2. $(L; \leq)$ is a lattice with

$$\sup\{a, b\} = a \vee b \quad \text{and} \quad \inf\{a, b\} = a \wedge b$$

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Proof:

1. reflexive by (L3), antisymmetric by (L2), transitive by (L1)
2. First show that $a \vee b \in \{a, b\}^u$ then show that $d \in \{a, b\}^u \implies (a \vee b) \leq d$. Easy by applying the (Li) to the suitable premises (Exercise).

Functions on Partial Orders

Let P be a partial order. A function $f : P \to P$ is

- monotone if for all $x, y \in P$:

$$x \leq y \implies f(x) \leq f(y)$$

- continuous if for each directed subset $M \subseteq L$:

$$f(\bigsqcup M) = \bigsqcup f(M)$$

Lemma: Continous functions are monotone.
Proof: Exercise

Knaster-Tarski Fixpoint Theorem

Let L be a complete lattice and $f : L \to L$ be monotone. Then

$$\bigwedge\{x \in L \mid f(x) \leq x\}$$

is the least fixpoint of f. (The dual holds analogously.)

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is the least fixpoint of f. (The dual holds analogously.)

Proof: Let $R := \{x \in L \mid f(x) \leq x\}$ be the set of elements of which f is *reductive*. Let $x \in R$. Consider $z = \bigwedge R$. z exists, because L is complete. $z \leq x$ because z is a lower bound of x. By monotonicity, $f(z) \leq f(x)$. Because $x \in R$, $f(z) \leq x$. Thus, $f(z)$ is also a lower bound of R. Thus, $f(z) \leq y$ for all $y \in R$. Because z is the greatest lower bound of R, $f(z) \leq z$, thus $z \in R$. By monotonicity, $f(f(z)) \leq f(z)$. Hence, $f(z) \in R$. Because z is a lower bound of R, $z \leq f(z)$ and $z = f(z)$.

Finite Lattices Are Complete

Associativity allows us to write sequences of joins unambiguously without brackets. One can show (by induction) that

$$\bigvee\{a_1, \ldots, a_n\} = a_1 \vee \cdots \vee a_n$$

for $\{a_1, \ldots, a_n\} \in L$, $n \geq 2$. Thus, for any finite, non-empty subset $F \in L$, $\bigvee$ and $\bigwedge$ exist.

Thus, every finite lattice bounded (has a greatest and least element) with

$$\top = \bigvee L \qquad \bot = \bigwedge L$$

Finally, becuase finite lattices have $\bot$ ($\top$), it exists $\bigvee \emptyset$ ($\bigwedge \emptyset$):

$$\bot = \bigvee \emptyset \qquad \top = \bigwedge \emptyset$$

Hence, finite lattices are complete.

Fixpoint by Iteration (Kleene)

Let L be a complete lattice, $f : L \to L$ a monotone function, and $\alpha := \bigsqcup_{i \geq 0} f^i(\bot)$.

1. If α is a fixpoint, it is the least fixpoint.
2. If f is continuous, α is a fixpoint.

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Let L be a complete lattice, $f : L \to L$ a monotone function, and $\alpha := \bigsqcup_{i \geq 0} f^i(\bot)$.

1. If α is a fixpoint, it is the least fixpoint.
2. If f is continuous, α is a fixpoint.

Proof: First, α exists because L is a lattice.

1. Assume $\beta = f(\beta)$ is a fixpoint of f. By definition, $\bot \leq \beta$ and because f is monotone, for all i: $f^i(\bot) \leq f^i(\beta) = \beta$. Hence, β is an upper bound on $M = \{\bot, f(\bot), \dots\}$. Because α is the least upper bound of M, we have $\alpha \leq \beta$. Hence, if α is a fixpoint, it is the least.
2. $$\begin{aligned} f(\alpha) = f(\textstyle\bigsqcup_{i \geq 0} f^i(\bot)) &= \textstyle\bigsqcup_{i \geq 0} f(f^i(\bot)) \ f \text{ continuous} \\ &= \textstyle\bigsqcup_{i \geq 1} f^i(\bot) \\ &= \textstyle\bigsqcup_{i \geq 0} f^i(\bot) \qquad \text{because } \forall i. \bot \leq f^i(\bot) \\ &= \alpha \end{aligned}$$

Remark: The theorem also holds for complete partial orders in which only every ascending chain must have a least upper bound.

Fixpoints in Complete Lattices