Exercise Sheet 2

Sebastian Hack
Jan Reineke
Christian Hammer
Tomasz Dudziak

Abstractions, concretizations, and Galois connections

Exercise 2.1: 3 points

Following set **SignConst** is an extension of the lattice **Sign** given in the lecture.

$$\mathbf{SignConst} = \{\bot, \top, \{+\}, \{-\}, \{-,0\}, \{0,+\}, \{-,+\}\} \cup \mathbb{Z}$$

The meaning of elements of **SignConst** will be given by a following function $\gamma\colon \mathbf{SignConst} \to \mathcal{P}(\mathbb{Z})$.

$$\begin{aligned}
\gamma(\top) &= \mathbb{Z} \\
\gamma(\bot) &= \emptyset \\
\gamma(\{+\}) &= \{x \in \mathbb{Z}\colon\ x > 0\} \\
\gamma(\{-\}) &= \{x \in \mathbb{Z}\colon\ x < 0\} \\
\gamma(\{-,0\}) &= \gamma(\{-\}) \cup \{0\} \\
\gamma(\{0,+\}) &= \gamma(\{+\}) \cup \{0\} \\
\gamma(\{-,+\}) &= \mathbb{Z} \setminus \{0\} \\
\gamma(x) &= \{x\}
\end{aligned}$$

In a way, **SignConst** combines constant propagation with rule-of-signs analysis. The ordering $\sqsubseteq$ is defined using the function γ as:

$$\forall x, y \in \mathbf{SignConst}.\ x \sqsubseteq y \iff \gamma(x) \subseteq \gamma(y)$$

Your task is to do the following.

1. Draw a Hasse diagram for **SignConst**. You can use dots ("...") to avoid writing down all elements of the set $\mathbb{Z}$. Determine the height of **SignConst** where height is defined as:

$$\text{height}(P) := \max\{|X|\colon X \subseteq P \wedge \forall a, b \in X.\ a \leq b \vee a \geq b\}$$

2. Define the abstract addition operator for **SignConst**. That is, write a formula for $[\![e_1 + e_2]\!]^\#$ as a function of $[\![e_1]\!]^\#$ and $[\![e_2]\!]^\#$. Your operator should fulfill the correctness condition:

$$\gamma([\![e_1 + e_2]\!]^\#) \supseteq \{n_1 + n_2\colon n_1 \in \gamma([\![e_1]\!]^\#) \wedge n_2 \in \gamma([\![e_2]\!]^\#)\}$$

Exercise 2.2: 3 points

Suppose that $(C, \leq)$ and $(A, \sqsubseteq)$ are complete lattices and $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection between them. Prove that for any $x \in C$, $\alpha(x) = \bigsqcap\{a \in A\colon x \leq \gamma(a)\}$.

Exercise 2.3: 3 points

Prove that if $(C, \leq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (A, \sqsubseteq)$ is a Galois connection, then $\forall l_1, l_2 \in A.\ \gamma(l_1 \sqcap l_2) = \gamma(l_1) \wedge \gamma(l_2)$

Exercise 2.4: 3 points

For following definitions of P and $\gamma\colon P \to \mathcal{P}(\mathbb{Z})$ determine whether there exists a function $\alpha\colon \mathcal{P}(\mathbb{Z}) \to P$ and a relation $\sqsubseteq$ such that $(\mathcal{P}(\mathbb{Z}), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (P, \sqsubseteq)$ is a Galois connection.

1. Let A be a non-empty finite subset of $\mathbb{Z}$. We define $P = \mathcal{P}(A) \cup \{\bot\}$ and

$$\gamma(X) = (\mathbb{Z} \setminus A) \cup X$$
$$\gamma(\bot) = \emptyset$$

   Note: Be careful not to confuse $\emptyset \in P$ with $\bot \in P$.

2. $P = \mathbb{Z} \cup \{\bot, \top\}$ with $\gamma(n) = \mathbb{Z} \setminus \{n\}$, $\gamma(\bot) = \emptyset$, $\gamma(\top) = \mathbb{Z}$.

3. $P = \{\text{even}, \text{positive}, \bot, \top\}$ and

$$\gamma(\text{even}) = \{n \in \mathbb{Z} \colon 2|n\}$$
$$\gamma(\text{positive}) = \{n \in \mathbb{Z} \colon n > 0\}$$
$$\gamma(\bot) = \emptyset$$
$$\gamma(\top) = \mathbb{Z}$$