Lattices

Slides follow Davey and Priestley: *Introduction to Lattices and Order*

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Partial Orders

Let P be a set. A binary relation $\sqsubseteq$ on P is a partial order iff it is:

1. reflexive: $(\forall x \in P)\, x \sqsubseteq x$
2. transitive: $(\forall x, y, z \in P)\, x \sqsubseteq y \wedge y \sqsubseteq z \implies x \sqsubseteq z$
3. antisymmetric: $(\forall x, y \in P)\, x \sqsubseteq y \wedge y \sqsubseteq x \implies x = y$

An element $\bot$ with $\bot \sqsubseteq x$ for all $x \in P$ is called bottom element. It is unique by definition. Analogously, $\top$ is called top element, if $\top \sqsupseteq x$ for all $x \in P$.

Duality

Let P an ordered set. The dual P^D of P is obtained by defining $x \sqsubseteq y$ in P^D whenever $y \sqsubseteq x$ in P.

For every statement Φ about P there is a dual statement Φ^D about P^D. It is obtained from P by exchanging $\sqsubseteq$ by $\sqsupseteq$.

If Φ is true for all ordered sets, Φ^D is also true for all ordered sets.

Hasse Diagrams

A partial order $(P, \sqsubseteq)$ is typically visualized by a Hasse diagram:

- Elements of P are points in the plane
- If $x \sqsubseteq z$, then z is drawn above x.
- If $x \sqsubseteq z$, and there is no y with $x \sqsubseteq y \sqsubseteq z$, then x and z are connected by a line

The Hasse diagram of the dual of P is obtained by "flipping" the one of P by 180 degrees.

Upper and Lower Bounds

Let $(P, \sqsubseteq)$ be a partial ordered set and let $S \subseteq P$. An element $x \in P$ is a lower bound of S, if $x \sqsubseteq s$ for all $s \in S$. Let

$$S^\ell = \{x \in P \mid (\forall s \in S)\, x \sqsubseteq s\}$$

be the set of all lower bounds of the set S. Analogously:

$$S^u = \{x \in P \mid (\forall s \in S)\, x \sqsupseteq s\}$$

Note: $\emptyset^u = \emptyset^\ell = P$.

If S^ℓ has a greatest element, this element is called the greatest lower bound and is written $\inf S$. (Dually for least upper bound and $\sup S$.) The greatest lower bound only exists, iff there is a $x \in P$ such that

$$(\forall y \in P)\, (((\forall s \in S)\, s \sqsupseteq y) \iff x \sqsupseteq y)$$

Lattices

The order-theoretic definition

Let P be an ordered set.

- If $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for every pair $x, y \in P$ then P is called a lattice.
- If For every $S \subseteq P$, $\sup S$ and $\inf S$ exist, then P is called a complete lattice.

The Connecting Lemma

Let L be a lattice and let $a, b \in L$. The following statements are equivalent:

1. $a \sqsubseteq b$
2. $\inf\{a, b\} = a$
3. $\sup\{a, b\} = b$

Lattices

The algebraic definition

We now view L as an algebraic structure $(L; \sqcup, \sqcap)$ with two binary operators

$$x \sqcup y := \sup\{x, y\} \qquad x \sqcap y := \inf\{x, y\}$$

Theorem: $\sqcup$ and $\sqcap$ satisfy for all $a, b, c \in L$:

$$
\begin{array}{lll}
(L1) & (a \sqcup b) \sqcup c = a \sqcup (b \sqcup c) & \text{associativity} \\
(L1)^D & (a \sqcap b) \sqcap c = a \sqcap (b \sqcap c) & \\
(L2) & a \sqcup b = b \sqcup a & \text{commutativity} \\
(L2)^D & a \sqcap b = b \sqcap a & \\
(L3) & a \sqcup a = a & \text{idempotency} \\
(L3)^D & a \sqcap a = a & \\
(L4) & a \sqcup (a \sqcap b) = a & \text{absorption} \\
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(L4) & a \sqcup (a \sqcap b) = a & \text{absorption} \\
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\end{array}$$

Proof: (L2) is immediate because $\sup\{x, y\} = \sup\{y, x\}$. (L3), (L4) follow from the connection lemma. (L1) for exercise. The dual laws come by duality.

Lattices

From the algebraic to the order-theoretic definition

Let $(L; \sqcup, \sqcap)$ be a set with two operators satisfying $(L1)$–$(L4)$ and $(L1)^D$–$(L4)^D$

Theorem:

1. Define $a \sqsubseteq b$ on L if $a \sqcup b = b$. Then, $\sqsubseteq$ is a partial oder
2. With $\sqsubseteq$, $(L; \sqsubseteq)$ is a lattice with

$$\sup\{a, b\} = a \sqcup b \quad \text{and} \quad \inf\{a, b\} = a \sqcup b$$

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Proof:

1. reflexive by (L3), antisymmetric by (L2), transitive by (L1)
2. First show that $a \sqcup b \in \{a, b\}^u$ then show that $d \in \{a, b\}^u \implies (a \sqcup b) \sqsubseteq d$. Easy by applying the (Li) to the suitable premises (Exercise).

Finite Lattices

Associativity allows us to write sequences of joins unambiguously without brackets. One can show (by induction) that

$$\bigsqcup\{a_1, \ldots, a_n\} = a_1 \sqcup \cdots \sqcup a_n$$

for $\{a_1, \ldots, a_n\} \in L$, $n \geq 2$. Thus, for any finite, non-empty subset $F \in L$, $\bigsqcup$ and $\bigsqcap$ exist.

Thus, every finite lattice bounded (as a greatest and least element) with

$$\top = \bigsqcup L \qquad \bot = \bigsqcap L$$

Further, every finite lattice is complete because

$$\bot = \bigsqcup \emptyset \qquad \top = \bigsqcap \emptyset$$

Knaster-Tarski Fixpoint Theorem

Let L be a complete lattice and $f : L \to L$ be monotone. Then

$$\bigsqcap \{x \in L \mid f(x) \sqsubseteq x\}$$

is the least fixpoint of f. (The dual holds analogously).

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Proof: Let $R := \{x \in L \mid f(x) \sqsubseteq x\}$ be the set of elements of which f is *reductive*. Let $x \in R$. Consider $z = \bigsqcap R$. z exists, because L is complete. $z \sqsubseteq x$ because z is a lower bound of x. By monotonicity, $f(z) \sqsubseteq f(x)$. Because $x \in R$, $f(z) \sqsubseteq x$. Thus, $f(z)$ is also a lower bound of R. Thus, $f(z) \sqsubseteq y$ for all $y \in R$. Because z is the greatest lower bound of R, $f(z) \sqsubseteq z$, thus $z \in R$. By monotonicity, $f(f(z)) \sqsubseteq f(z)$. Hence, $f(z) \in R$. Because z is a lower bound of R, $z \sqsubseteq f(z)$ and $z = f(z)$.

Fixpoint by Iteration

Let L be a complete finite lattice and $f : L \to L$ be monotone. Hence, every chain $a_1 \sqsubseteq \cdots \sqsubseteq a_n$ stabilizes, i.e. there is a $k < n$ such that $a_k = a_{k+1}$

1. It holds $\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \ldots$
2. $d = f^{n-1}(\bot) = f^n(\bot)$ is the smallest element d' with $f(d') \sqsubseteq d'$

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Proof: (1) exercise. (2): d exists because of (1) and the assumption that every ascending chain stabilizes. Consider another $d' \sqsupseteq d$ with $f(d') \sqsubseteq d'$.
We show (by induction) that for every $i \in \mathbb{N}$ there is $f^i(\bot) \sqsubseteq d'$.
Let $i = 0$: $\bot \sqsubseteq d'$ holds. Now assume $f^{i-1}(\bot) \sqsubseteq d'$. Then

$$f^i(\bot) = f(f^{i-1}(\bot)) \sqsubseteq f(d') \sqsubseteq d'$$